Taylor Polynome

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Inhalt

Über Taylor

·Brook Taylor

·Satz von Taylor

·Bestandteil der Analysis

·Entwickelte die Taylor Polynome

Taylor Polynom

- **Eine beliebige Funktion** $f(x)$
- **Für** x **einen bestimmten Wert** a
- $f(a)$
- **Diese Funktion durch ein Polynom (x, x², x³,...) möglichst genau umschreiben**
- **Es gilt:**

$$T(a) = f(a)$$

&

$$\lim_{x \to a} T(a) = \lim_{x \to a} f(a)$$

Berechnung

- **Simpelster Weg: Eine Tangente am Punkt** a

$$T(x) = m * x + b$$

$$m = f'(a) \qquad x = x - a \qquad b = f(a)$$

$$T(x) = f'(a) * (x - a) + f(a)$$

Beweis

$$T(x) = f'(a) * (x - a) + f(a)$$

- **Für den Unterschied von T zu f gilt somit**

$$R(x) \equiv f(x) - T(x)$$

- **Daraus folgt**

$$\lim_{x \to a} \frac{R(x)}{(x-a)} = 0$$

Berechnung 2.

- **Genauer wird die Taylor Funktion, wenn man sie durch eine quadratische Funktion beschreibt**

$$T(x) = a + bx + cx^2$$

$$a = f(a) \quad b = f'(a) \quad c = \frac{1}{2} f''(a) \quad x = (x - a)$$

$$T(x) = f(a) + f'(a) * (x - a) + \frac{1}{2} f''(a) * (x - a)^2$$

Beweis 2.

$$T(x) = f(a) + f'(a) * (x-a) + \frac{1}{2} f''(a) * (x-a)^2$$

- **Für den Unterschied von T zu f gilt somit**

$$R(x) \equiv f(x) - T(x)$$

- **Daraus folgt**

$$\lim_{x \to a} \frac{R(x)}{(x-a)^2} = 0$$

Beschreibt die Funktion genauer

Allgemein

- **Aus den beiden Polynomen lässt sich nun eine generelle Formel n-ten Grades aufstellen**

$$T(x) = f(a) + \frac{1}{1} f'(a) * (x-a)^1 + \frac{1}{2} f''(a) * (x-a)^2 + \ldots + \frac{1}{n!} f^n(a) * (x-a)^n$$

- **Die Restgleichung löst sich dann folgend auf**

$$\lim_{x \to a} \frac{R(x)}{(x-a)^n} = 0$$

Allgemein

- **Zuletzt folgt die generelle Gleichung**

$$T(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{k}(a) * (x-a)^{k}$$

Beispiel

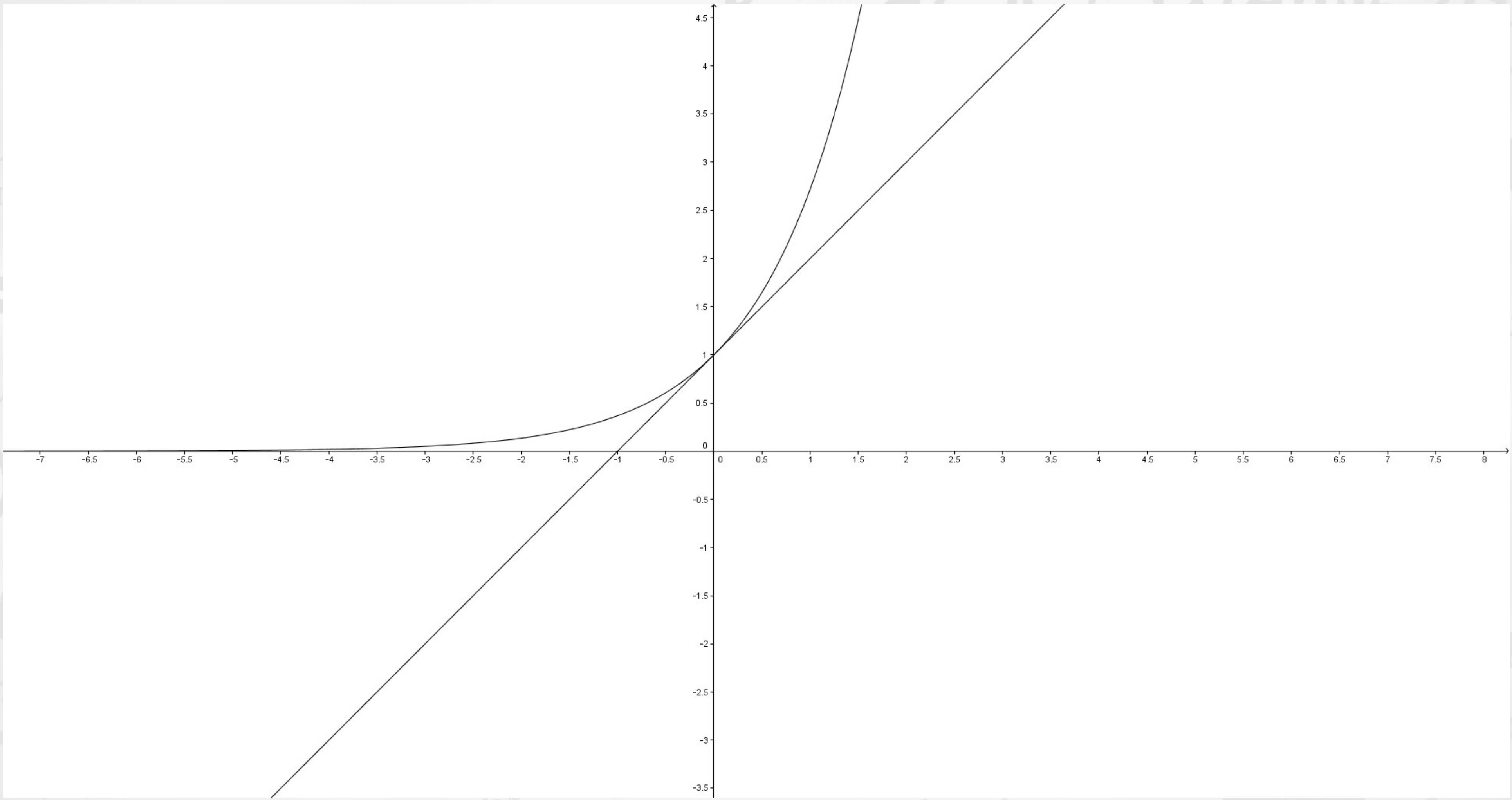
- **Das Beispiel ist** $f(x)=e^{x}$

- **Gebildet wir das 5te Taylor Polynom für** $a=0$

$$T(x)=f(a)+f'(a)*(x-a)+\frac{1}{2}f''(a)*(x-a)^2+\frac{1}{3!}f'''(a)*(x-a)^3+\frac{1}{4!}f''''(a)*(x-a)^4+\frac{1}{5!}f'''''(a)*(x-a)^5$$

$$T(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{5040}x^5$$

Grafik



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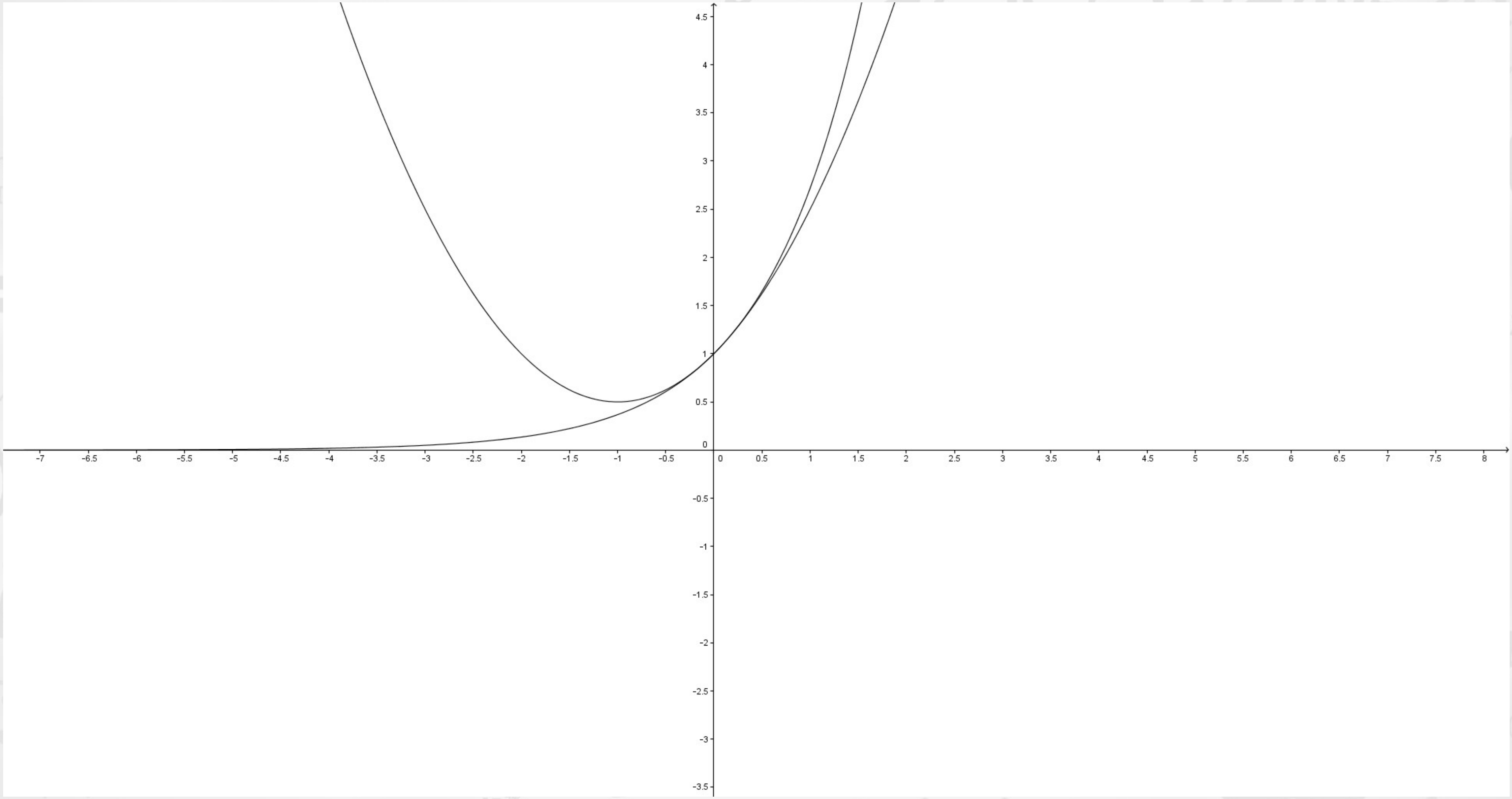

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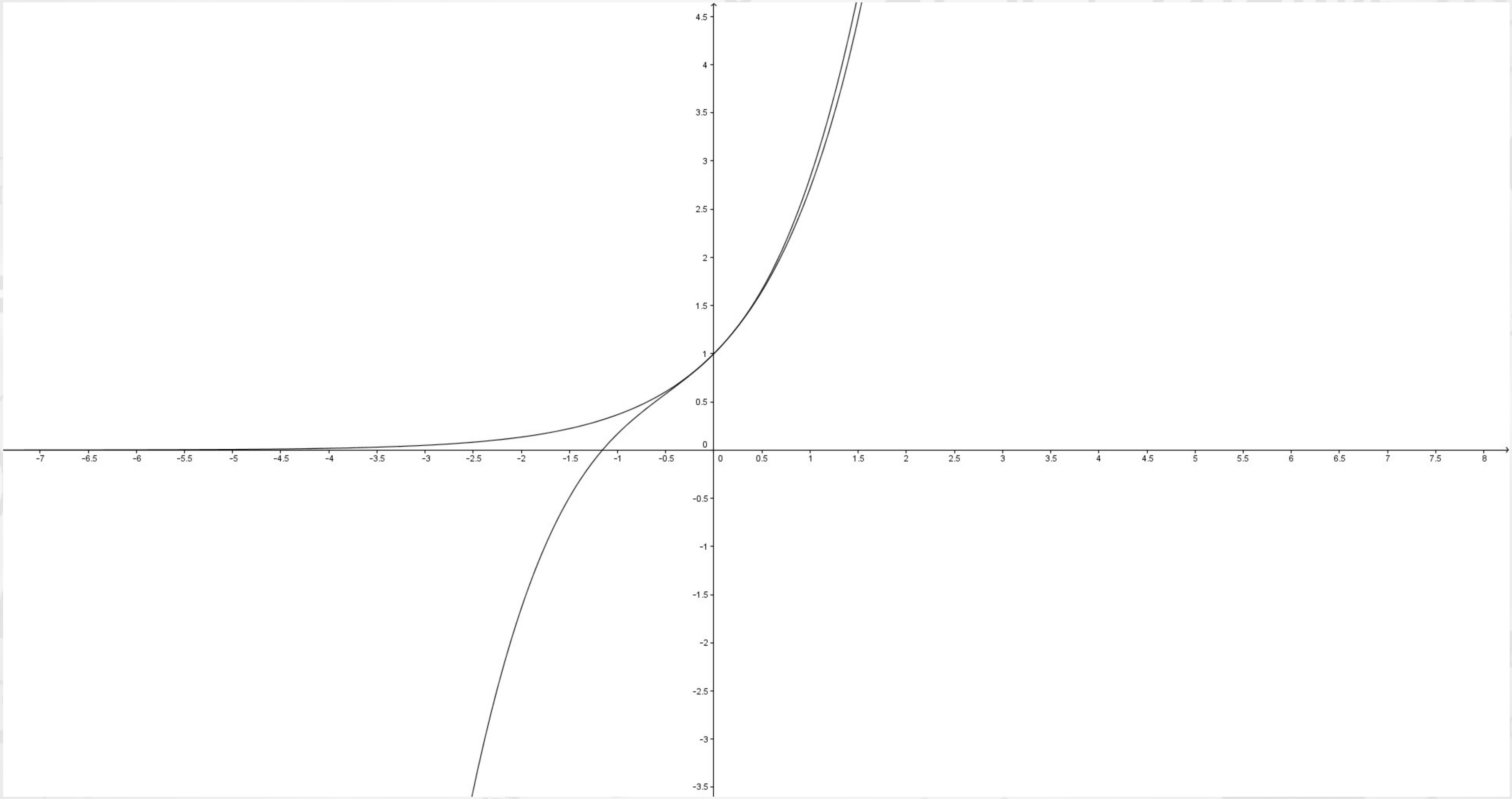

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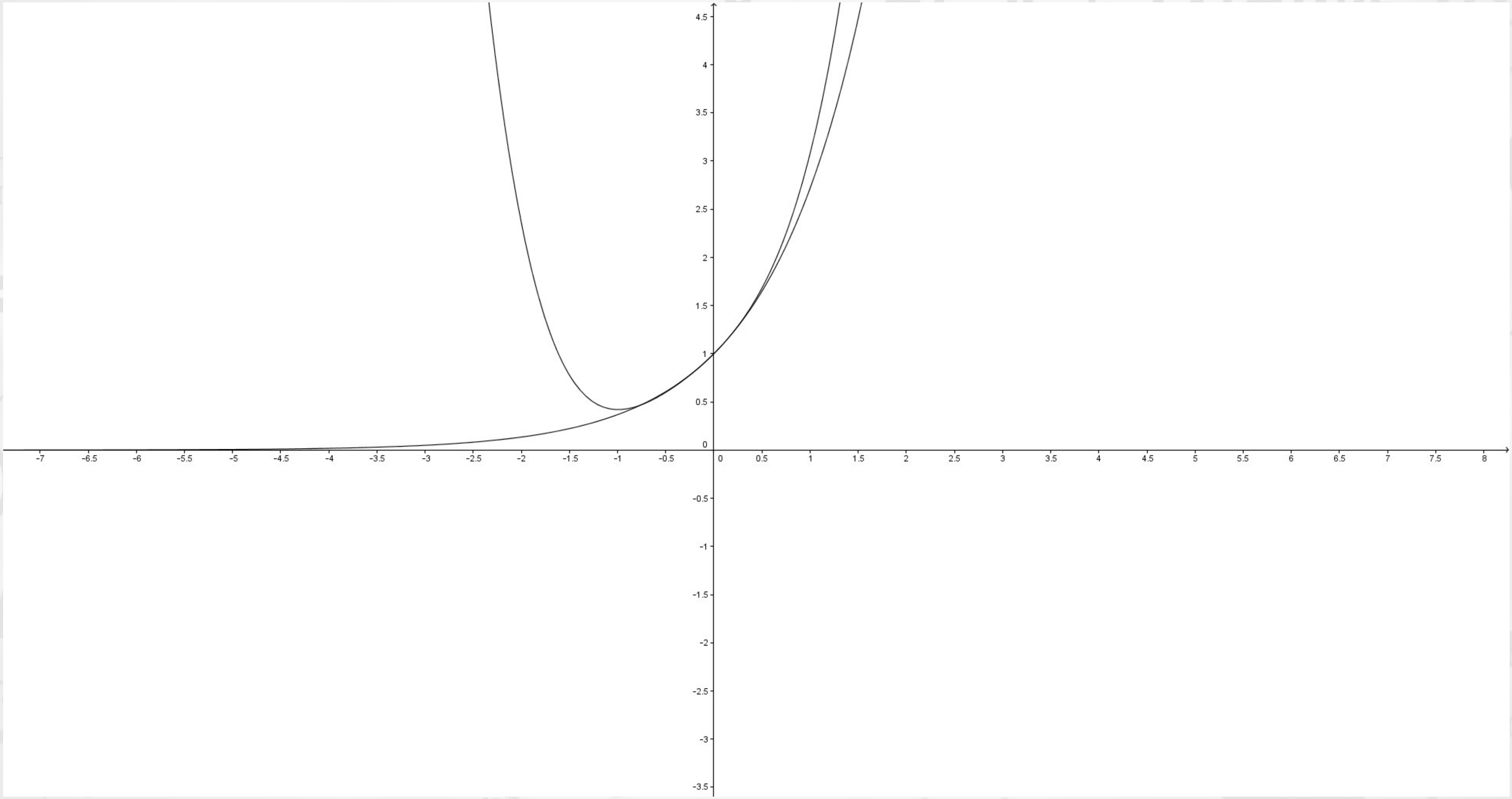

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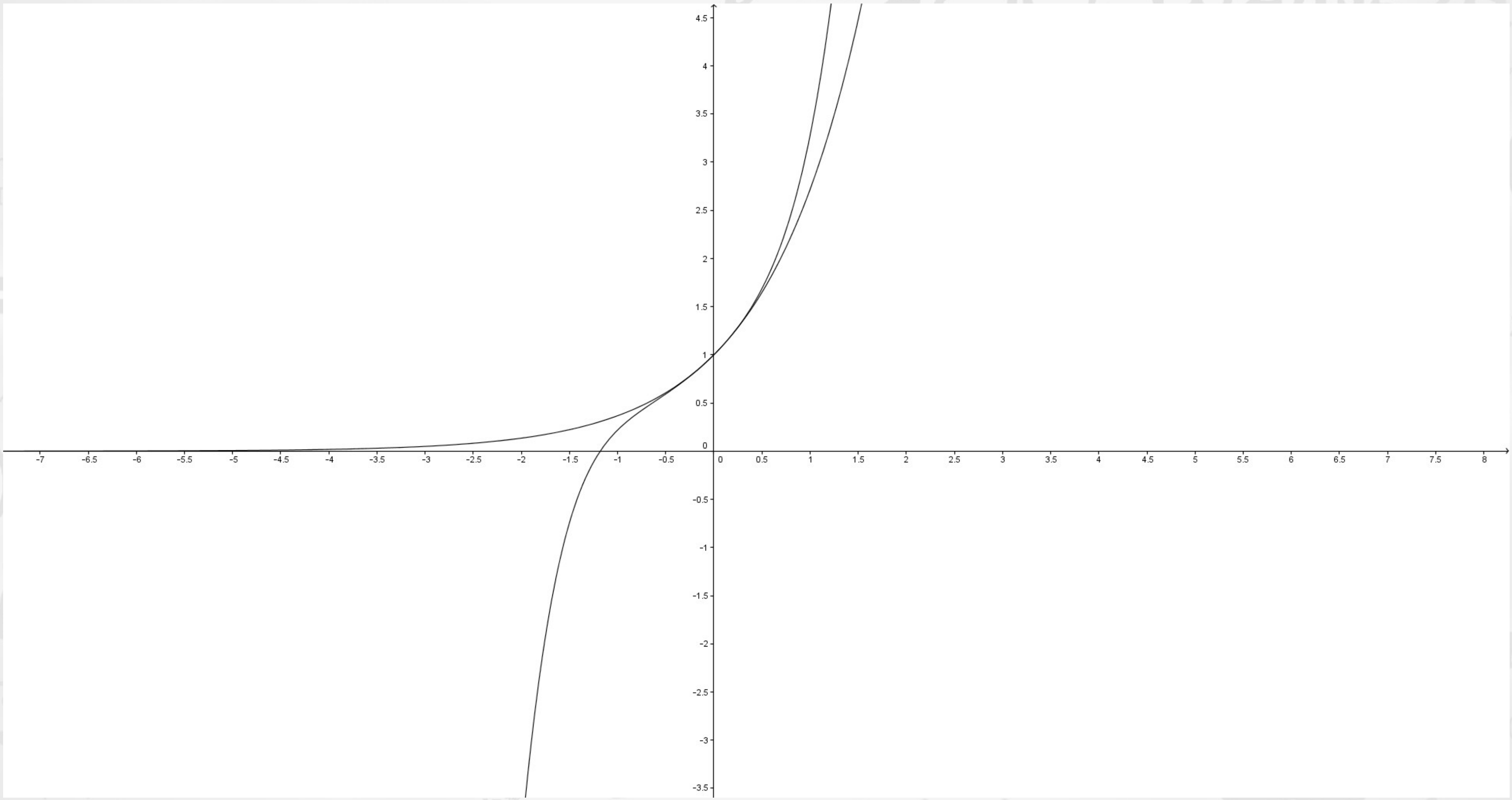


Beispiel

- **Das Beispiel ist** $f(x)=\sin(x)$

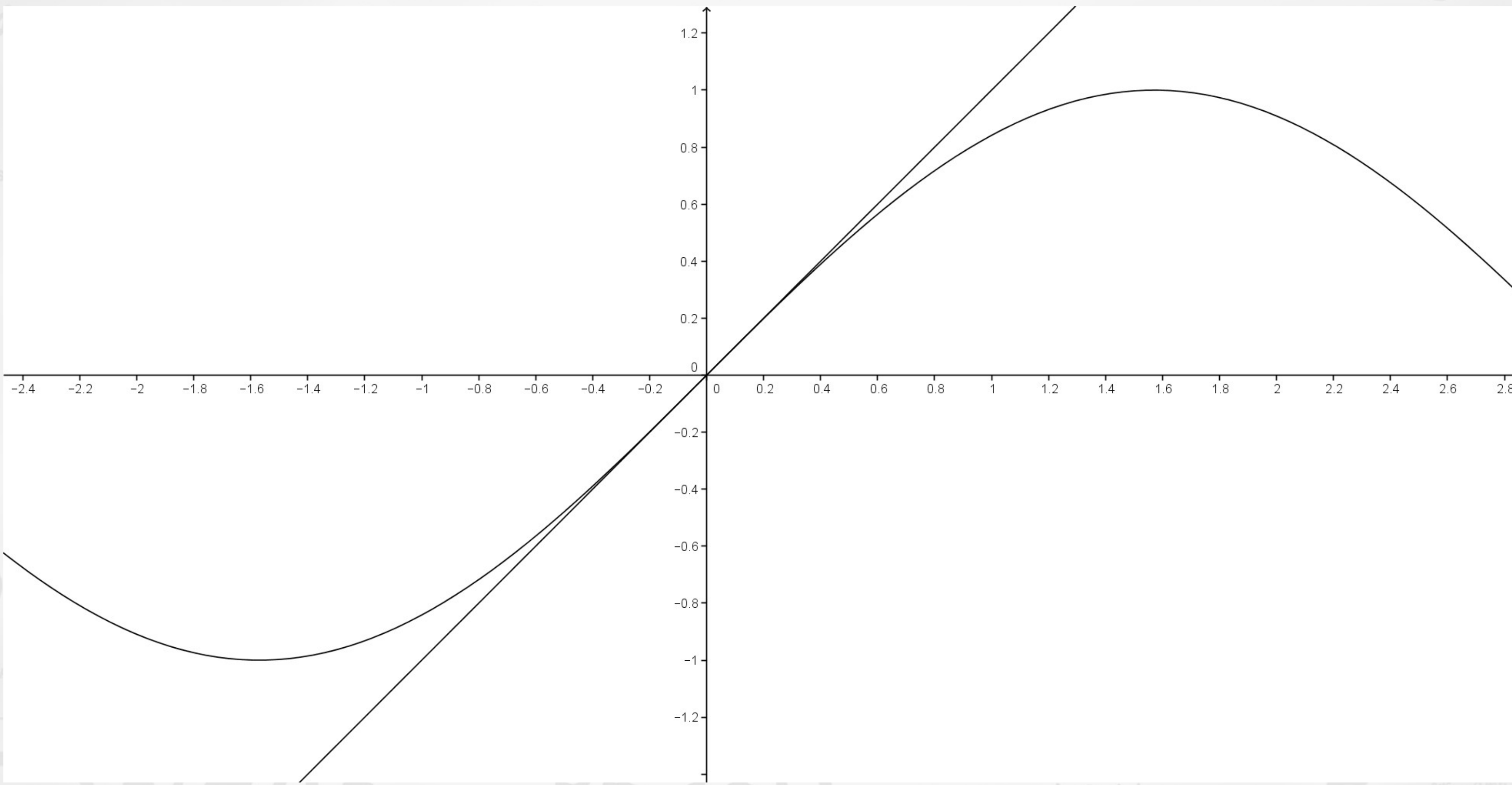
- **Gebildet wir das 10te Taylor Polynom für** $a=0$

$$T(x)=f(a)+f'(a)*(x-a)+\frac{1}{2}f''(a)*(x-a)^2+\frac{1}{3!}f'''(a)*(x-a)^3+\frac{1}{4!}f''''(a)*(x-a)^4+\frac{1}{5!}f'''''(a)*(x-a)^5$$

$$\frac{1}{6!}f''''''(a)*(x-a)^6+\frac{1}{7!}f'''''''(a)*(x-a)^7+\frac{1}{8!}f''''''''(a)*(x-a)^8+\frac{1}{9!}f'''''''''(a)*(x-a)^9+\frac{1}{10!}f''''''''''(a)*(x-a)^{10}$$

$$T(x)=0+x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\frac{1}{362880}x^9$$

Grafik



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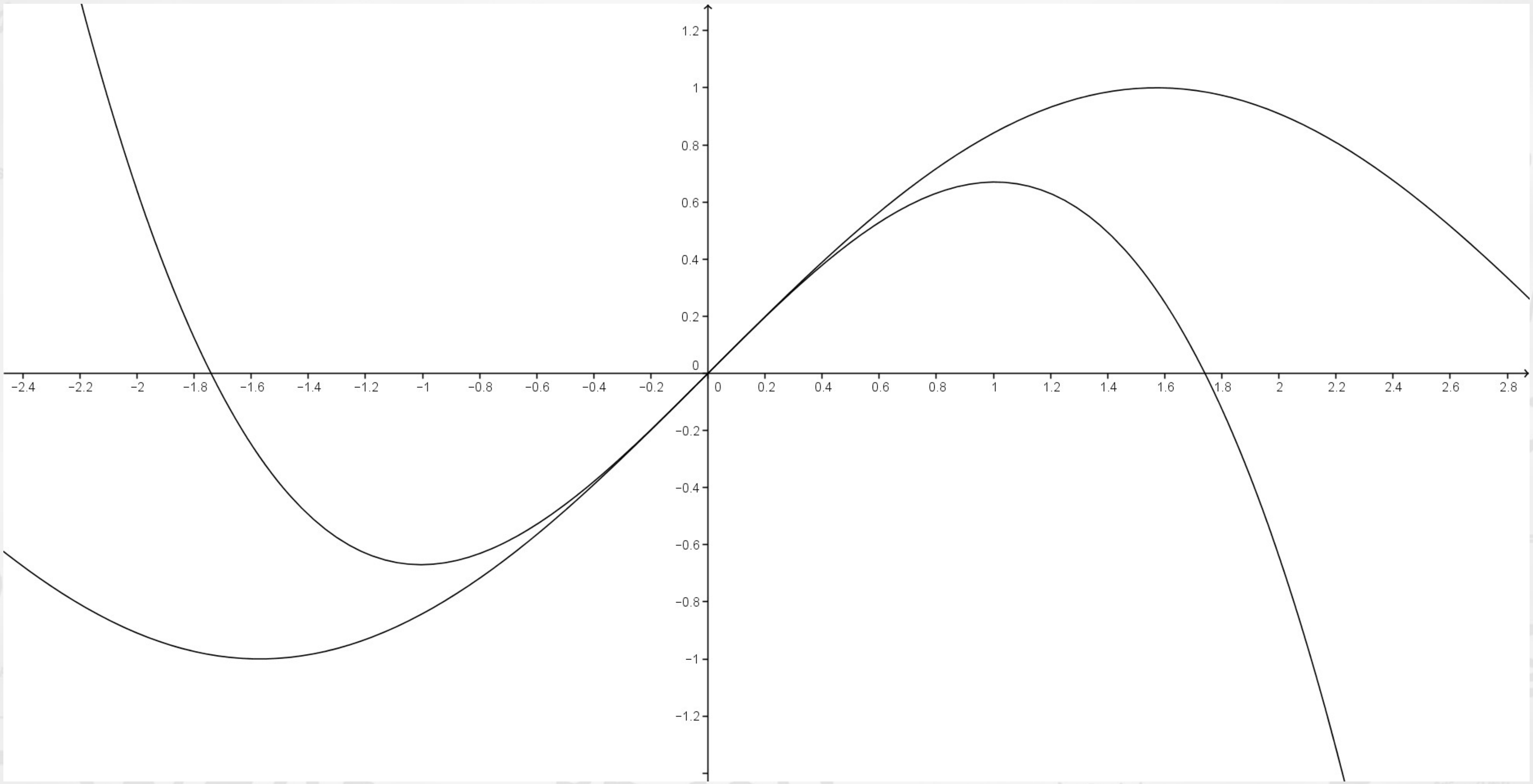
Grafik



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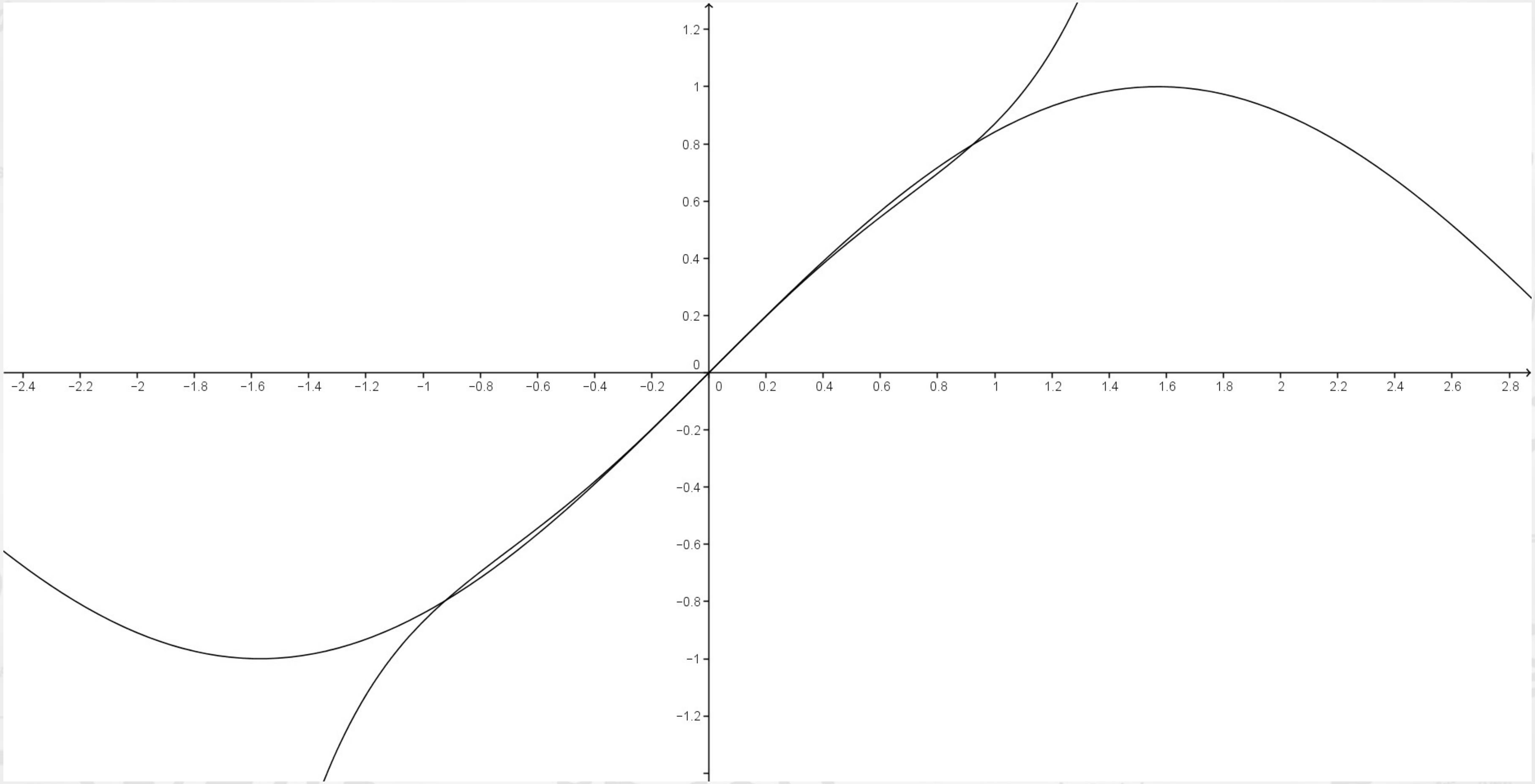

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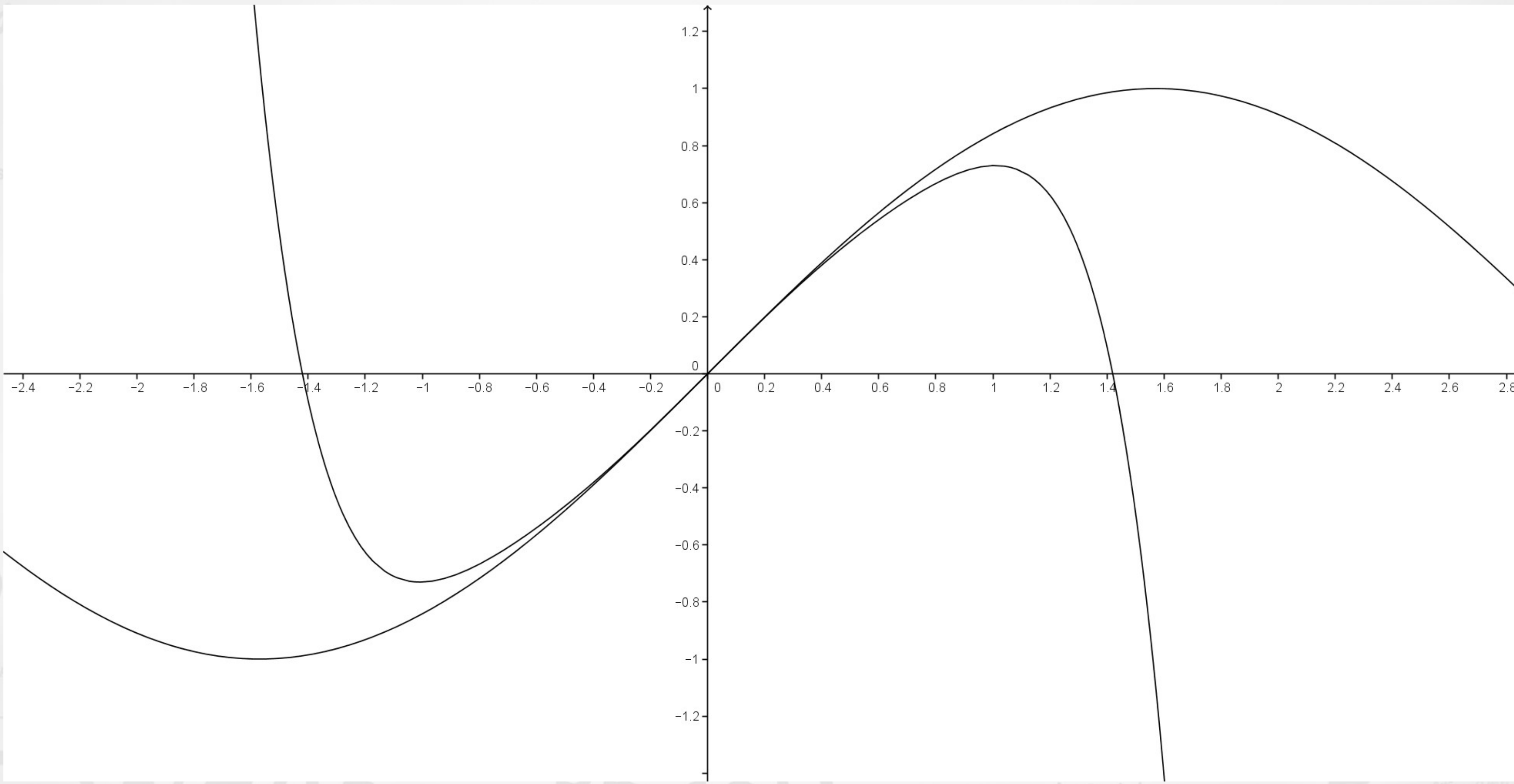

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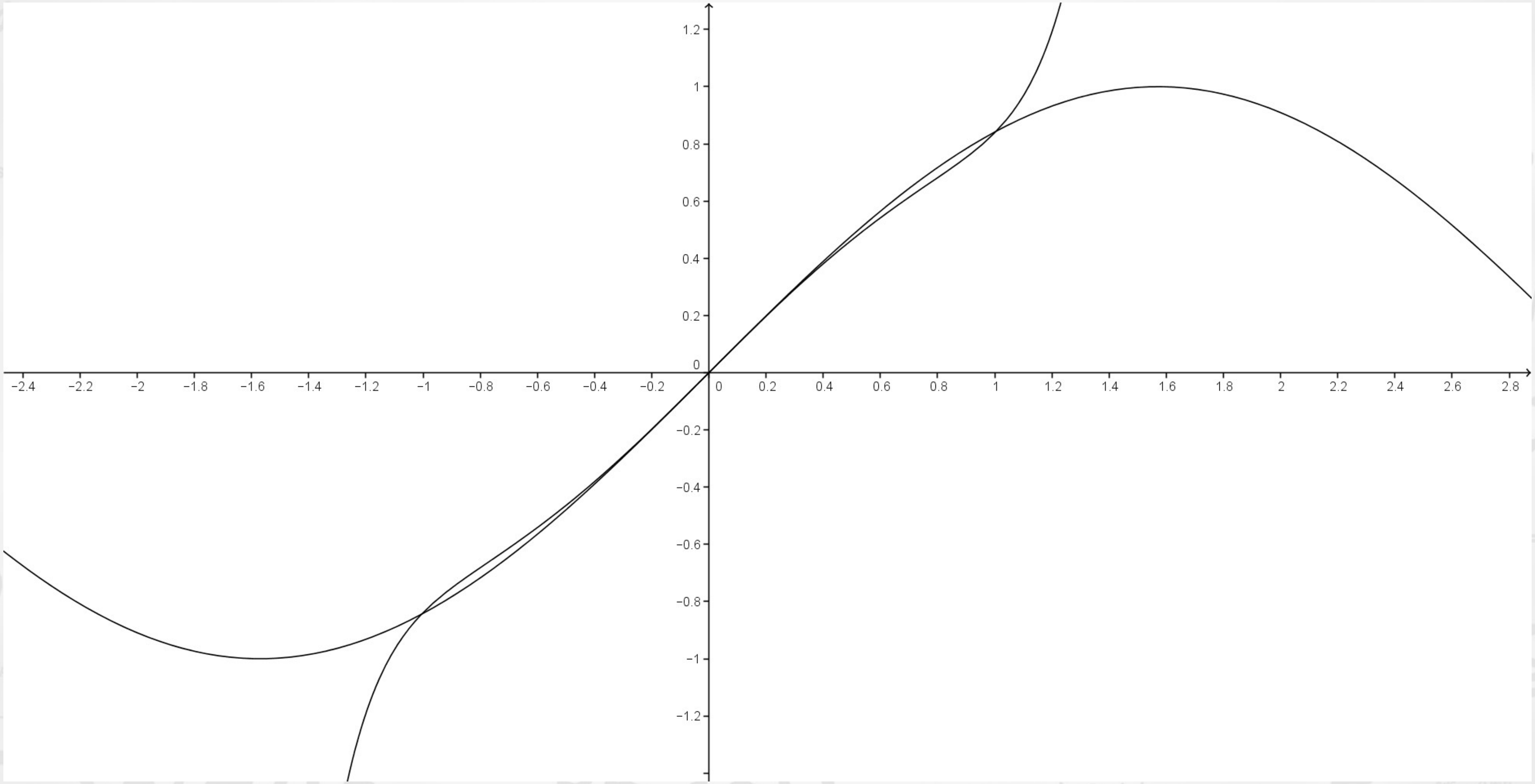

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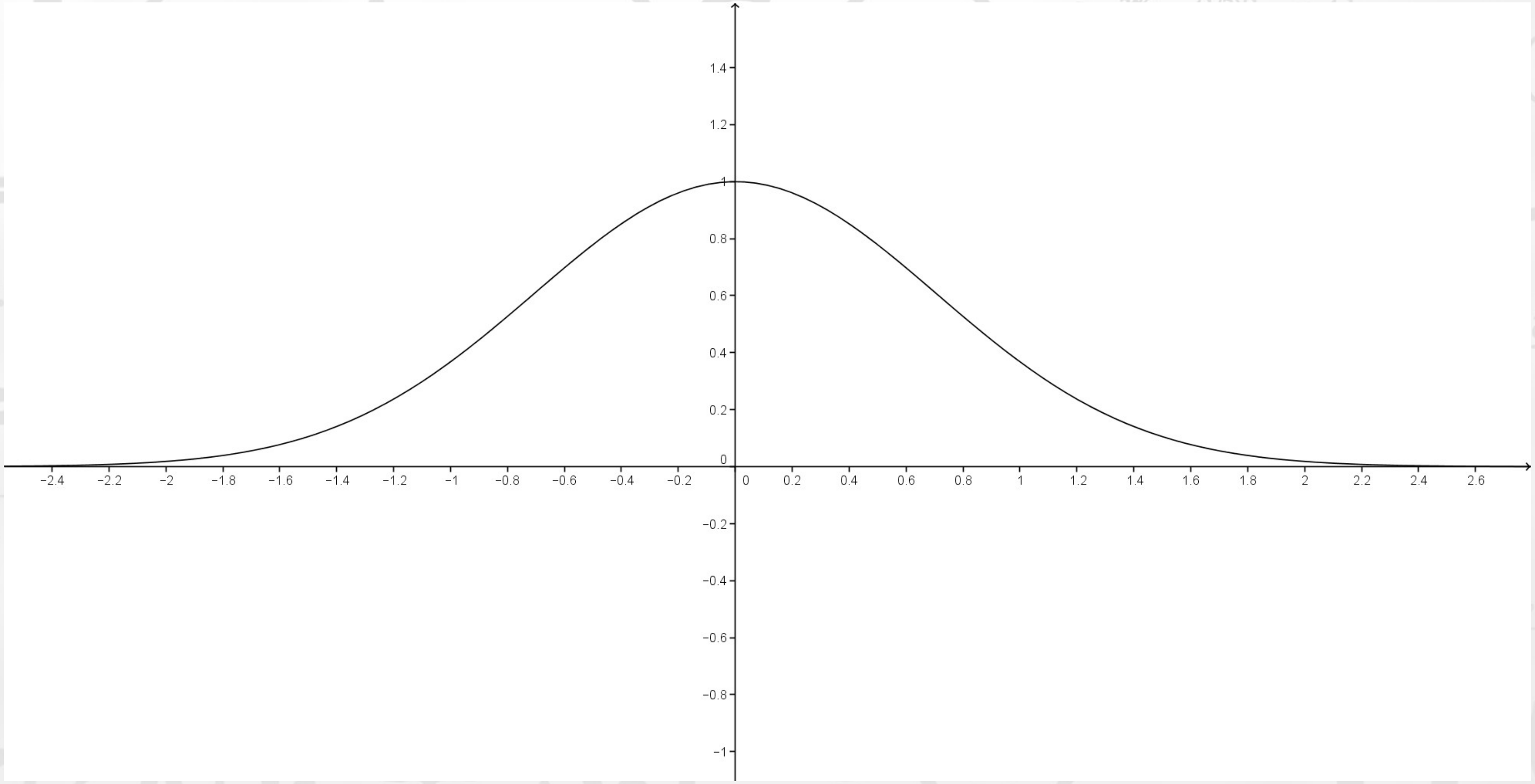
Zusatz

- **Die Funktion $f(x) = e^{-x^2}$ soll durch ein Taylor Polynom beschrieben werden**
- **Selbe Vorgehensweise**



Rechnung

$$f(x)=e^{-x^2}$$

$$f'(x)=-2x*e^{-x^2}$$

$$f''(x)=(-2+4x^2)*e^{-x^2}$$

$$f'''(x)=(12x-8x^3)*e^{-x^2}$$

$$f''''(x)=(12-48x^2+16x^4)*e^{-x^2}$$

$$f(0)=1$$

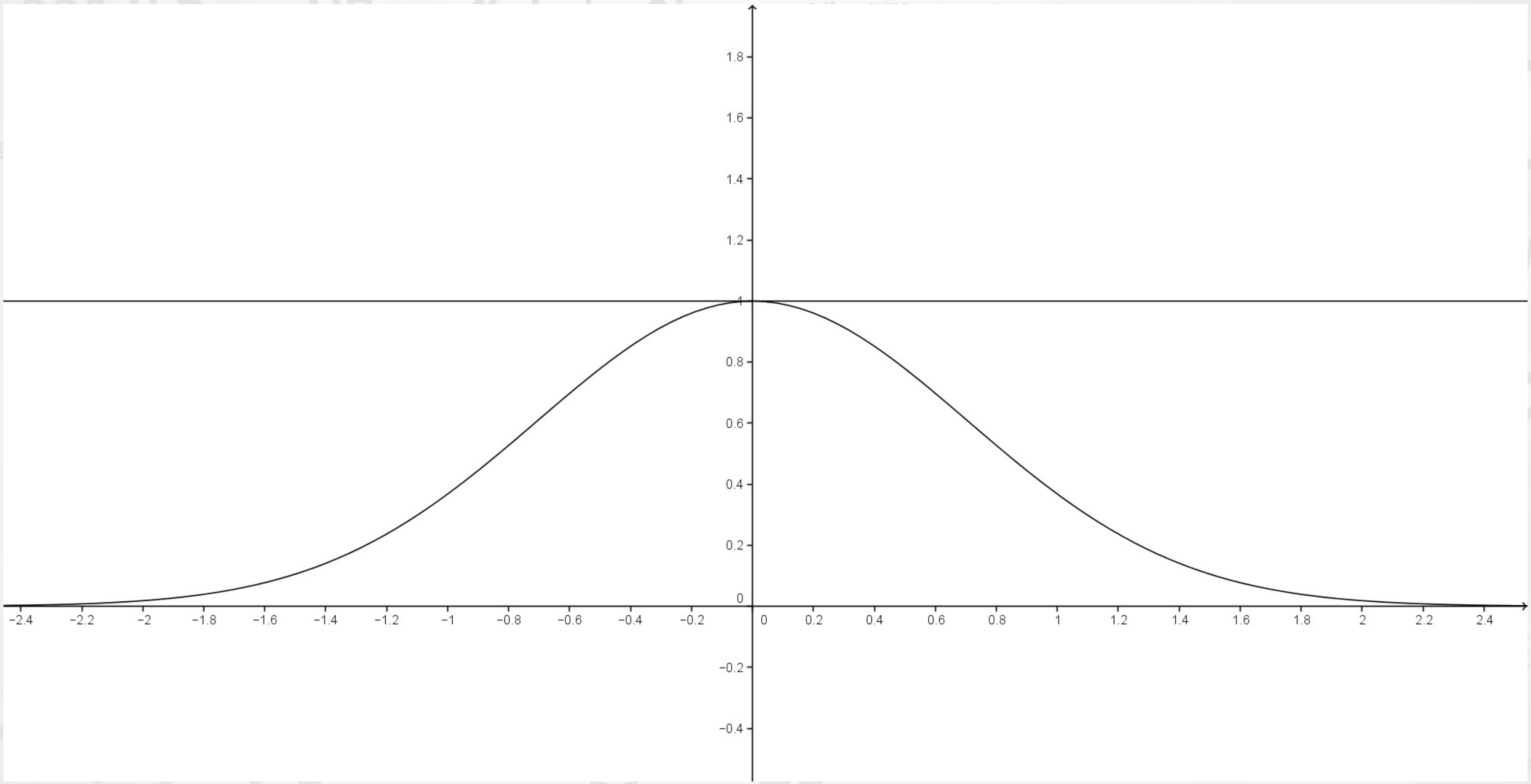
$$f'(0)=0$$

$$f''(0)=-2$$

$$f'''(0)=0$$

$$f''''(0)=12$$

Grafik



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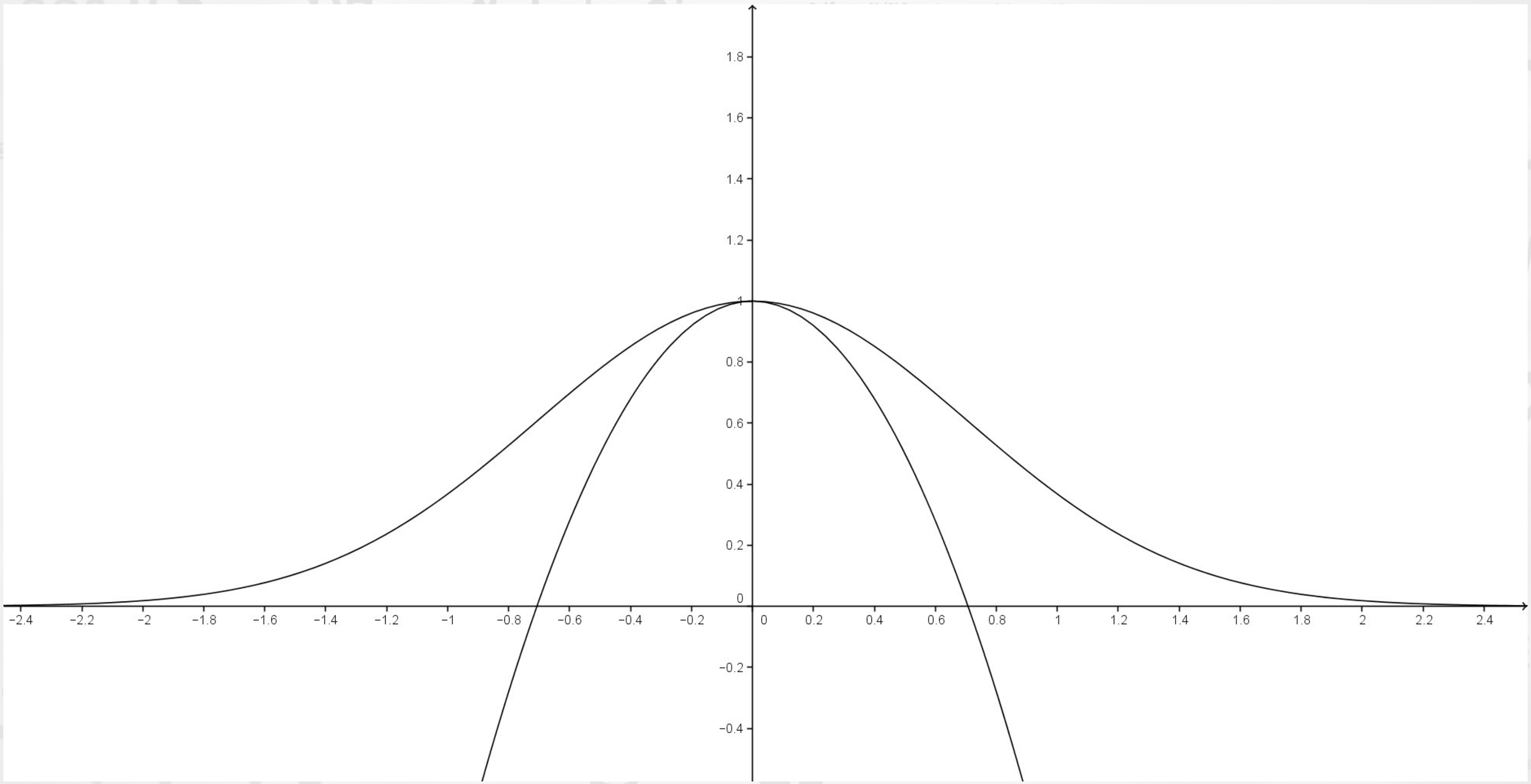

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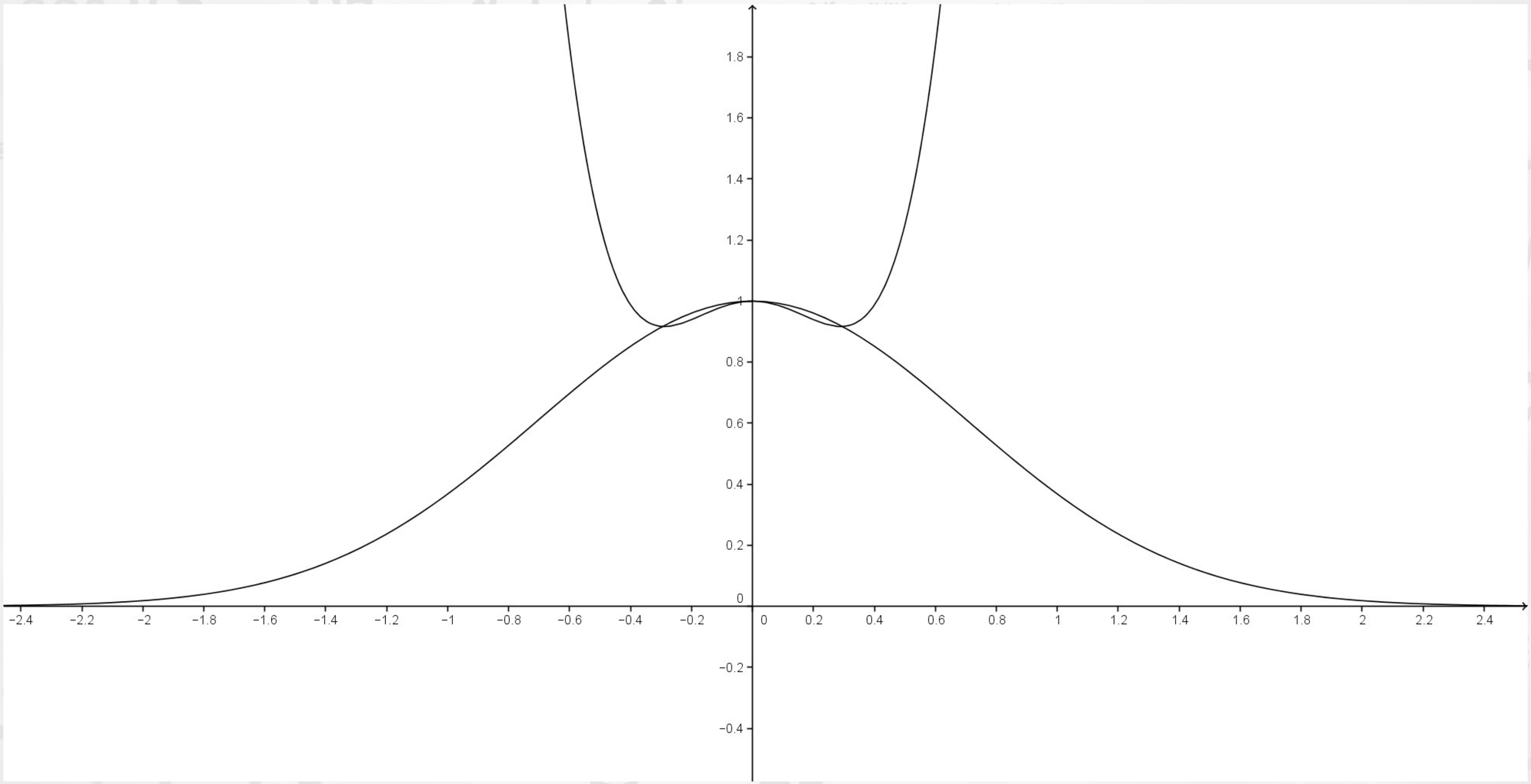
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Wichtigstes

- **Ein Taylor Polynom umschreibt eine Funktion nur, und ist nur in der Nähe der Bildungsstelle a zu gebrauchen**

$$T(x)=f(a)+\frac{1}{1}f'(a)*(x-a)^1+\frac{1}{2}f''(a)*(x-a)^2+\ldots+\frac{1}{n!}f^n(a)*(x-a)^n$$

$$T(x)=\sum_{k=0}^{n}\frac{1}{k!}f^k(a)*(x-a)^k$$

Quellen

- http://de.wikipedia.org/wiki/Taylor-Formel
- http://www.pohlig.de/Mathematik/Taylor/taylor.htm

Fotos:

- http://upload.wikimedia.org/wikipedia/commons/e/eb/Brook_Taylor.jpeg

- Grafiken generiert durch GeoGebra